JOURNAL OF COMPUTATIONAL PHYSICS 104, 129–139 (1993)

A Perturbational h^4 Exponential Finite Difference Scheme for the Convective Diffusion Equation

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Received September 19, 1990; revised March 18, 1992

A perturbational h^4 compact exponential finite difference scheme with diagonally dominant coefficient matrix and upwind effect is developed for the convective diffusion equation. Perturbations of second order are exerted on the convective coefficients and source term of an h^2 exponential finite difference scheme proposed in this paper based on a transformation to eliminate the upwind effect of the convective diffusion equation. Four numerical examples including one- to three-dimensional model equations of fluid flow and a problem of natural convective heat transfer are given to illustrate the excellent behavior of the present exponential schemes. Besides, the h^4 accuracy of the perturbational scheme is verified using double precision arithmetic. © 1993 Academic Press, Inc.

1. INTRODUCTION

The convective diffusion equation is of primary importance in such fields as fluid mechanics and heat transfer. In customary finite difference treatment of the equation, as reviewed by Anderson *et al.* [2], the diffusion terms expressed by second derivatives are approximated by central difference, and the obstacle to giving a satisfactory scheme is usually considered to be the presence of the convective terms involving first derivatives, which result in the so-called upwind effect. If the first derivatives are approximated by central difference, the resulting finite difference equation has an accuracy of second order, but the associated coefficient matrix may fail to be diagonally dominant when the convective coefficients become large, compared with the grid size. As is well known, diagonal dominance is usually required for the discrete approximations pertaining to the convective diffusion equation to guarantee the maximum principle of the finite difference scheme: solutions in the field domain are bounded by the values given on the boundary, when the equation contains no source term. In contrast, when the first derivatives are approximated by upwind difference, the resulting finite difference equation is first-order accurate, although the corresponding matrix is diagonally dominant unconditionally. To obtain a scheme satisfactory in both accuracy and stability, we may resort to finite difference approximations involving exponential coefficients.

The exponential finite difference scheme of the convective diffusion equation was first introduced by Allen and Southwell [1] in approximating vorticity transport equation for obtaining their numerical solutions for steady viscous flow past a circular cylinder. According to Allen and Southwell, steady two-dimensional convective diffusion equation

$$2A\frac{\partial\phi}{\partial x}+2B\frac{\partial\phi}{\partial y}=\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+S. \tag{1.1}$$

(In numerical treatment, the convective coefficients A and B and source term S may be regarded as known function of x and y) is separated into two equations

$$2A_0\frac{\partial\phi}{\partial x}=\frac{\partial^2\phi}{\partial x^2}+R_0 \tag{1.2}$$

$$2B_0\frac{\partial\phi}{\partial y}=\frac{\partial^2\phi}{\partial y^2}+S_0-R_0, \tag{1.3}$$

where A, B, and S in the discrete subdomain illustrated in Fig. 1 are replaced by their nodal values A_0, B_0, and S_0; R_0 is constant. Using the solutions of Eq. (1.2) and Eq. (1.3) as expansions along the x-line 301 and y-line 402, respectively, we can express R_0 in terms of nodal values ϕ_3, ϕ_0, and ϕ_1, or in ϕ_4, ϕ_0, and ϕ_2. Equating the values of R_0 gives a finite difference equation on a five-point stencil 01234, which was

0021-9991/93 $5.00

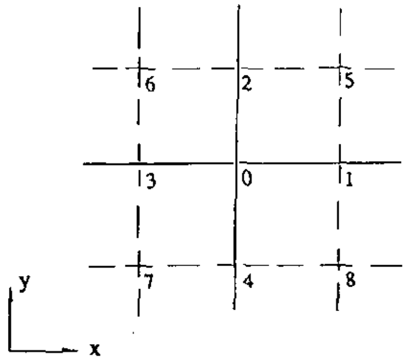

FIG. 1. The nine-point stencil for finite differences schemes.

subsequently examined by Dennis [5, 6] who revealed the accuracy of second order and pointed out the important property of being diagonally dominant of the equation. Afterwards, Spalding [11] and Roscoe [10] described some methods which appear to be essentially the same as that given by Allen and Southwell [11].

The method of Dennis [5] is somewhat general. The convective diffusion equation (1.1) is separated into the two equations

$$2A\frac{\partial\phi}{\partial x}=\frac{\partial^2\phi}{\partial x^2}+R \tag{1.4}$$

$$2B\frac{\partial\phi}{\partial y}=\frac{\partial^2\phi}{\partial y^2}+S-R, \tag{1.5}$$

where A, B, and S remain functions of x and y instead of being replaced by their nodal values as in the Allen and Southwell method. Along the x-line, $A(x, y)=A(x, y_0)$ and $R(x, y)=R(x, y_0)$, Eq. (1.4) becomes an ordinary differential equation, of which the solution for ϕ is a function of x only. By means of a standard technique to eliminate the term of first derivative in linear ordinary differential equations of second order, we put

$$\phi(x, y_0)=\psi(x)\,\theta(x) \tag{1.6}$$

$$\psi(x)=\exp\left[\int A(x, y_0)\,dx\right] \tag{1.7}$$

to transform Eq. (1.4) into

$$\left[A^2-\frac{dA(x, y_0)}{dx}\right]\theta=\frac{d^2\theta}{dx^2}+\frac{R(x, y_0)}{\psi(x)} \tag{1.8}$$

which is a diffusion equation with no convective term. Approximating the second derivative in Eq. (1.8) by central difference and combining the resulting discrete equation of θ with transformation (1.6) gives an expression of $R(x_0, y_0)$ in terms of nodal values ϕ_3, ϕ_0 and ϕ_1. Similar treatment of Eq. (1.5) along the y-line 402 gives an expression of $R(x_0, y_0)$ in values ϕ_4, ϕ_0, and ϕ_2. Finally, a finite difference scheme on five-point stencil 01234 is reached by equating the values of $R(x_0, y_0)$. The scheme is second-order accurate and has a matrix of diagonal dominance after improvement [8].

Dennis and Hudson [7] held that the advantage of Dennis's method over the Allen and Southwell approximation lies in the fact that Dennis's can be improved in accuracy more conveniently by adding difference corrections in the way of Fox [4]. The modified exponential finite difference approximations of fourth-order accuracy [7, 8] with corresponding polynomial counterparts [9] have compact structure involving only values of ϕ at the nine points centered on a typical point (x_0, y_0) of a two-dimensional square grid and can give very accurate results on comparatively coarse cell grids for the problems they chose as examples. Nevertheless, in their opinion [9], what seems to be a blemish in an otherwise perfect thing is that the associated matrixes for both the exponential and polynomial fourth-order accurate schemes may fail to be diagonally dominant, even for the one-dimensional case. In connection with this fact, Dennis and Hudson [9] have given a basic formula of h^3 accuracy by using an upwind approximation to a higher derivative which is always diagonally dominant and which may be corrected by adding a deferred correction to make it h^4. Even so, for the solutions thus obtained, no maximum principle is guaranteed, and non-physical spurious oscillations may occur.

More importantly, the upwinding property of the finite difference scheme should be examined. Inherent to convective diffusion phenomena, we have the following effect, often referred to as upwinding in the field of computational fluid mechanics and heat transfer [2]: the convective diffusion quantity ϕ is more strongly affected by the upwind ϕ than by the downwind ϕ; and for convection-dominated problems the downwind influence becomes negligible, compared with the upwind one. Concretely, in the one-dimensional finite difference scheme,

$$a_i\phi_i=a_{i-1}\phi_{i-1}+a_{i+1}\phi_{i+1}+S_{mi} \tag{1.9}$$

for the convective-diffusion equation

$$2A\frac{\partial\phi}{\partial x}=\frac{\partial^2\phi}{\partial x^2}+S \tag{1.10}$$

(where S_{mi} stands for a discrete source term), the upwinding property requires: (i) the downwind coefficient a_d (a_{i+1} if $A_i>0$, or a_{i-1} if $A_i\leqslant 0$) should be always less than the upwind coefficient a_u (a_{i-1} if $A_i>0$, or a_{i+1} if $A_i\leqslant 0$); and a_d will become negligible compared with a_u when the flow field is convection-dominated (A becomes very large). In the compact h^4 scheme developed by Dennis and Hudson [9], corresponding coefficients can be written as

$$a_{i-1}=\alpha_i^2+\frac{1}{12}\left(A_i^2h^2+\left.\frac{\partial^2 A}{\partial x^2}\right|_i h^3\right) \tag{1.11}$$

$$a_{i+1}=\beta_i^2+\frac{1}{12}\left(A_i^2h^2-\left.\frac{\partial^2 A}{\partial x^2}\right|_i h^3\right) \tag{1.12}$$

$$a_i=a_{i-1}+a_{i+1}, \tag{1.13}$$

where

$$\alpha_i=1+\frac{1}{2}A_ih-\frac{1}{6}\left.\frac{\partial A}{\partial x}\right|_i h^2 \tag{1.14}$$

$$\beta_i=1-\frac{1}{2}A_ih-\frac{1}{6}\left.\frac{\partial A}{\partial x}\right|_i h^2 \tag{1.15}$$

and h stands for the cell size. Here we only consider the simplest case of constant A. The coefficient expressions are thence reduced to

$$a_{i-1}=1+R_{ec}+\tfrac{1}{3}R_{ec}^2 \tag{1.16}$$

$$a_{i+1}=1-R_{ec}+\tfrac{1}{3}R_{ec}^2, \tag{1.17}$$

where $R_{ec}=A_ih$ is the cell Reynolds number, which is a basic dimensionless criterion in computational fluid mechanics [2]. It can be easyly appreciated that (i) the downwind coefficient a_d is always less than the upwind one a_u, but unfortunately (ii) the difference between a_d and a_u becomes less and less as R_{ec} increased, and eventually for great R_{ec} value, a_d becomes actually identical with a_u. This contradiction to upwind effect states that the compact h^4 scheme by Dennis and Hudson [9] is certainly not applicable for convection-dominated problems, which will be illustrated in detail in the numerical example on the fluid flow model equation.

The object of the present paper is to advance a method of giving exponential finite difference of second-order accuracy for the convective diffusion equation based on a general transformation to eliminate upwind effect and to improve the second-order accurate scheme with special care for preserving the upwind effect and the property of being diagonally dominant of the associated matrix while achieving compactness and higher order accuracy. Methods of this nature have not been considered previously. Numerical examples including one- to three-dimensional model equations of fluid flow and a problem of natural convective heat transfer are given to illustrate the behavior of the present exponential schemes. Besides, the fourth-order accuracy of the improved scheme is verified using double precision arithmetic.

2. THE UPWIND FUNCTION AND AN h^2 ACCURATE EXPONENTIAL SCHEME OF THE CONVECTIVE DIFFUSION EQUATION

By putting

$$\phi(x,y)=\psi(x,y)\,\theta(x,y), \tag{2.1}$$

Eq. (1.1) is transformed into

$$2A\left(\frac{\partial\psi}{\partial x}\theta+\psi\frac{\partial\theta}{\partial x}\right)+2B\left(\frac{\partial\psi}{\partial y}\theta+\psi\frac{\partial\theta}{\partial y}\right)$$
$$=\left(\frac{\partial^2\psi}{\partial x^2}\theta+2\frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial x}+\psi\frac{\partial^2\theta}{\partial x^2}\right)$$
$$+\left(\frac{\partial^2\psi}{\partial y^2}\theta+2\frac{\partial\psi}{\partial y}\frac{\partial\theta}{\partial y}+\psi\frac{\partial^2\theta}{\partial y^2}\right)+S. \tag{2.2}$$

If

$$A\psi=\frac{\partial\psi}{\partial x},\qquad B\psi=\frac{\partial\psi}{\partial y}, \tag{2.3}$$

Eq. (2.2) becomes a diffusion equation,

$$\left[\left(A^2-\frac{\partial A}{\partial x}\right)+\left(B^2-\frac{\partial B}{\partial y}\right)\right]\theta$$
$$=\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}+\frac{S}{\psi}. \tag{2.4}$$

In Eq. (2.4), the convective coefficients A, B are equally located, where involving convective coefficients is merely a diffusion source without preferred orientation; consequently there is no upwind effect retained. Hence the upwind effect is completely embodied by the function defined by Eq. (2.3); we call ψ the upwind function of the convective diffusion equation (1.1) accordingly. From the standpoint of pure analysis, ψ does not exist unconditionally. Yet fortunately, in finite difference treatment, ψ is required only in special subdomains such as the x-line 301 and y-line 402 as shown in Fig. 1. Under such circumstances, we have ψ unconditionally expressed as

$$\psi=\exp\left[\int(A\,dx+B\,dy)\right]. \tag{2.5}$$

Approximating the diffusion equation (2.4) by standard three-point central differencing gives

$$\frac{1}{h_1^2}(\theta_{i+1,j}-2\theta_{i,j}+\theta_{i-1,j}) + \frac{1}{h_2^2}(\theta_{i,j+1}-2\theta_{i,j}+\theta_{i,j-1}) = \left[\left(A_{i,j}^2-\frac{A_{i+1,j}-A_{i-1,j}}{2h_1}\right) + \left(B_{i,j}^2-\frac{B_{i,j+1}-B_{i,j-1}}{2h_2}\right)\right]\theta_{i,j}-\frac{S_{i,j}}{\psi_{i,j}}, \quad (2.6)$$

where h_1, h_2 are grid sizes in the x, y direction, respectively. In the subdomain concerned, the upwind function may be simply taken as

$$\psi=\exp(A_{i,j}x+B_{i,j}y). \quad (2.7)$$

Substituting $\phi_{i,j}/\psi_{i,j}$ for $\theta_{i,j}$ in Eq. (2.6), we obtain the following finite difference equation for ϕ:

$$\frac{1}{h_1^2}(e^{-A_{i,j}h_1}\phi_{i+1,j}-2\phi_{i,j}+e^{A_{i,j}h_1}\phi_{i-1,j}) + \frac{1}{h_2^2}(e^{-B_{i,j}h_2}\phi_{i,j+1}-2\phi_{i,j}+e^{B_{i,j}h_2}\phi_{i,j-1}) = \left[\left(A_{i,j}^2-\frac{A_{i+1,j}-A_{i-1,j}}{2h_1}\right) + \left(B_{i,j}^2-\frac{B_{i,j+1}-B_{i,j-1}}{2h_2}\right)\right]\phi_{i,j}-S_{i,j}. \quad (2.8)$$

Owing to the fact that the convective diffusion equation has a special solution $\phi=\text{const}$, the coefficient of the central point 0 of an accurate finite difference equation of the convective diffusion equation should be equal to the summation of the coefficients of the neighbouring points 1, 2, 3, and 4. In order to rectify the deviation caused by discretization, we modified Eq. (2.8) into the following form:

$$2\left[\frac{1}{h_1^2}\cosh(A_{i,j}h_1)+\frac{1}{h_2^2}\cosh(B_{i,j}h_2)\right]\phi_{i,j} = \frac{1}{h_1^2}(e^{-A_{i,j}h_1}\phi_{i+1,j}+e^{A_{i,j}h_1}\phi_{i-1,j}) + \frac{1}{h_2^2}(e^{-B_{i,j}h_2}\phi_{i,j+1}+e^{B_{i,j}h_2}\phi_{i,j-1}) + S_{i,j}. \quad (2.9)$$

Employing expansion techniques in which the exponentials are expressed in powers of their arguments, the modified differential equation, which is equivalent to the corresponding finite difference equation (see Anderson *et al.* [2]), of the above scheme is found as

$$2A\frac{\partial\phi}{\partial x}+2B\frac{\partial\phi}{\partial y}=\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+S+2E_xh_1^2 + 2E_yh_2^2+O(h_1^4,h_2^4), \quad (2.10)$$

where

$$E_x=-\frac{1}{3!}A^3\frac{\partial\phi}{\partial x}+\frac{1}{2!2!}A^2\frac{\partial^2\phi}{\partial x^2} -\frac{1}{3!}A\frac{\partial^3\phi}{\partial x^3}+\frac{1}{4!}\frac{\partial^4\phi}{\partial x^4} \quad (2.11)$$

$$E_y=-\frac{1}{3!}B^3\frac{\partial\phi}{\partial y}+\frac{1}{2!2!}B^2\frac{\partial^2\phi}{\partial y^2} -\frac{1}{3!}B\frac{\partial^3\phi}{\partial x^3}+\frac{1}{4!}\frac{\partial^4\phi}{\partial y^4} \quad (2.12)$$

which states that the accuracy of the scheme is of second order.

The property of being diagonally dominant of the scheme is obvious. Excellent resolution ability to abrupt change in function value is illustrated in the numerical solution of the Burgers equation in Section 6 of this paper. The present h^2 accurate scheme is the simplest in representative form in the category of exponential scheme initiated by Allen and Southwell [1].

3. THE BASIC PERTURBATIONAL h^4 ACCURATE EXPONENTIAL SCHEME

In line with the previous section, for one dimensional convective diffusion equation

$$2A\frac{\partial\phi}{\partial x}=\frac{\partial^2\phi}{\partial x^2}+S \quad (3.1)$$

we have exponential h^2 accurate scheme,

$$\frac{2}{h^2}\cosh(A_ih)\,\phi_i = \frac{1}{h^2}(e^{-A_ih}\phi_{i+1}+e^{A_ih}\phi_{i-1})+S_i, \quad (3.2)$$

with modified differential equation

$$2A\frac{\partial\phi}{\partial x}=\frac{\partial^2\phi}{\partial x^2}+S+2E_xh^2+O(h^4), \quad (3.3)$$

where h stands for the grid size; E_x is expressed by Eq. (2.11).

As this scheme possesses excellent properties, it would be best to preserve its basic structure while improving its accuracy. Supposing there are perturbations of second order exerted on the convective coefficient and source term, the scheme becomes

$$\frac{1}{h^2}\cosh(A_{pi}h)\,\phi_i = \frac{1}{h^2}\left(e^{-A_{pi}h}\phi_{i+1} + e^{A_{pi}h}\phi_{i-1}\right) + S_{pi}, \tag{3.4}$$

where

$$A_p = A + \Delta A_2 \cdot h^2 \tag{3.5}$$

$$S_p = S + 2\Delta S_2 \cdot h^2, \tag{3.6}$$

which is equivalent to differential equation

$$2A_p\frac{\partial\phi}{\partial x} = \frac{\partial^2\phi}{\partial x^2} + S_p + 2E_x h^2 + O(h^4). \tag{3.7}$$

It can be seen that an h^4 accuracy is achieved if only

$$\Delta A_2\frac{\partial\phi}{\partial x} = \Delta S_2 + E_x + O(h^2). \tag{3.8}$$

From Eq. (3.7), we have

$$\frac{\partial^2\phi}{\partial x^2} = 2A\frac{\partial\phi}{\partial x} - S + O(h^2) \tag{3.9}$$

$$\frac{\partial^3\phi}{\partial x^3} = 2\left(\frac{\partial A}{\partial x} + 2A^2\right)\frac{\partial\phi}{\partial x} - 2AS - \frac{\partial S}{\partial x} + O(h^2) \tag{3.10}$$

$$\frac{\partial^4\phi}{\partial x^4} = \left(8A^3 + 12A\frac{\partial A}{\partial x} + 2\frac{\partial^2 A}{\partial x^2}\right)\frac{\partial\phi}{\partial x} - 4\left(\frac{\partial A}{\partial x} + A^2\right)S - 2A\frac{\partial S}{\partial x} - \frac{\partial^2 S}{\partial x^2} + O(h^2) \tag{3.11}$$

which substituted into Eq. (3.8) give expressions for the perturbational values

$$\Delta A_2 = \frac{1}{12}\left(2A\frac{\partial A}{\partial x} + \frac{\partial^2 A}{\partial x^2}\right) + O(h^2) \tag{3.12}$$

$$\Delta S_2 = \frac{1}{24}\left[2\left(A^2 + 2\frac{\partial A}{\partial x}\right)S - 2A\frac{\partial S}{\partial x} + \frac{\partial^2 S}{\partial x^2}\right] + O(h^2). \tag{3.13}$$

Discrete expressions for the perturbational values may be found by approximating the derivatives in Eqs. (3.12) and (3.13), using central differences of second-order accuracy,

$$(\Delta A_2)_i = \frac{1}{12h^2}\left[(1 - A_i h)\,A_{i-1} - 2A_i + (1 + A_i h)\,A_{i+1}\right] \tag{3.14}$$

$$(\Delta S_2)_i = \frac{1}{24h^2}\left\{(1 + A_i h)\,S_{i-1} + 2\left[-1 + A_i^2 h^2 + (A_{i+1} - A_{i-1})h\right]S_i + (1 - A_i h)\,S_{i+1}\right\}. \tag{3.15}$$

Excellent stability of the original h^2 accurate exponential scheme is preserved in the perturbational h^4 accurate exponential scheme.

4. PERTURBATIONAL h^4 ACCURATE EXPONENTIAL SCHEME IN TWO-DIMENSIONAL SPACE

Similar to the one-dimensional case, if perturbations of second order are exerted on the convective coefficients and source term, the two-dimensional h^2 accurate exponential scheme (2.9) becomes

$$2\left[\frac{1}{h_1^2}\cosh(A_{pij}h_1) + \frac{1}{h_2^2}\cosh(B_{pij}h_2)\right]\phi_{ij} = \frac{1}{h_1^2}\left[\exp(A_{pij}h_1)\,\phi_{i-1,j} + \exp(-A_{pij}h_1)\,\phi_{i+1,j}\right] + \frac{1}{h_2^2}\left[\exp(B_{pij}h_2)\,\phi_{i,j-1} + \exp(-B_{pij}h_2)\,\phi_{i,j+1}\right] + S_{pij}, \tag{4.1}$$

where

$$A_p = A + \Delta A_2 \cdot h_1^2 \tag{4.2}$$

$$B_p = B + \Delta B_2 \cdot h_2^2 \tag{4.3}$$

$$S_p = S + 2(\Delta S_{2x} \cdot h_1^2 + \Delta S_{2y} \cdot h_2^2). \tag{4.4}$$

The modified differential equation corresponding to scheme (4.1) is

$$2A_p\frac{\partial\phi}{\partial x} + 2B_p\frac{\partial\phi}{\partial y} = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + S_p + 2E_x \cdot h_1^2 + 2E_y \cdot h_2^2 + O(h^4), \tag{4.5}$$

where E_x and E_y are expressed by Eqs. (2.11) and (2.12). If only

$$\Delta A_2 \cdot h_1^2 \frac{\partial \phi}{\partial x} + \Delta B_2 \cdot h_2^2 \frac{\partial \phi}{\partial y} = (\Delta S_{2x} + E_x) h_1^2 + (\Delta S_{2y} + E_y) h_2^2 + O(h^4), \quad (4.6)$$

the perturbational scheme is h^4 accurate.

From Eq. (4.5), we have

$$\frac{\partial^2 \phi}{\partial x^2} = 2A \frac{\partial \phi}{\partial x} - S_x \quad (4.7)$$

$$\frac{\partial^2 \phi}{\partial y^2} = 2B \frac{\partial \phi}{\partial y} - S_y, \quad (4.8)$$

where

$$S_x = S + \left(-2B \frac{\partial \phi}{\partial y} + \frac{\partial^2 \phi}{\partial y^2} \right) \quad (4.9)$$

$$S_y = S + \left(-2A \frac{\partial \phi}{\partial x} + \frac{\partial^2 \phi}{\partial x^2} \right). \quad (4.10)$$

Differentiating Eqs. (4.7) and (4.8) with respect to x and y, respectively, gives proper expressions of the derivatives $\partial^3\phi/\partial x^3$, $\partial^4\phi/\partial x^4$, $\partial^3\phi/\partial y^3$, and $\partial^4\phi/\partial y^4$. Substituting those expressions into Eq. (4.6) yields the perturbational values

$$\Delta A_2 = \frac{1}{12} \left(2A \frac{\partial A}{\partial x} + \frac{\partial^2 A}{\partial x^2} \right) + O(h^2) \quad (4.11)$$

$$\Delta B_2 = \frac{1}{12} \left(2B \frac{\partial B}{\partial y} + \frac{\partial^2 B}{\partial y^2} \right) + O(h^2) \quad (4.12)$$

$$\Delta S_{2x} = \frac{1}{24} \left[2 \left(A^2 + 2A \frac{\partial A}{\partial x} \right) S_x - 2AP_x + Q_x \right] + O(h^2) \quad (4.13)$$

$$\Delta S_{2y} = \frac{1}{24} \left[2 \left(B^2 + 2B \frac{\partial B}{\partial y} \right) S_y - 2BP_y + Q_y \right] + O(h^2), \quad (4.14)$$

where

$$P_x = \frac{\partial S}{\partial x} - 2B \frac{\partial^2 \phi}{\partial x \, \partial y} - 2 \frac{\partial B}{\partial x} \frac{\partial \phi}{\partial y} + \frac{\partial^3 \phi}{\partial x \, \partial y^2} \quad (4.15)$$

$$P_y = \frac{\partial S}{\partial y} - 2A \frac{\partial^2 \phi}{\partial x \, \partial y} - 2 \frac{\partial A}{\partial y} \frac{\partial \phi}{\partial x} + \frac{\partial^3 \phi}{\partial y \, \partial x^2} \quad (4.16)$$

$$Q_x = \frac{\partial^2 S}{\partial x^2} - 2B \frac{\partial^3 \phi}{\partial x^2 \, \partial y} - 4 \frac{\partial B}{\partial x} \frac{\partial^2 \phi}{\partial x \, \partial y} - 2 \frac{\partial^2 B}{\partial x^2} \frac{\partial \phi}{\partial y} + \frac{\partial^4 \phi}{\partial x^2 \, \partial y^2} \quad (4.17)$$

$$Q_y = \frac{\partial^2 S}{\partial y^2} - 2A \frac{\partial^3 \phi}{\partial y^2 \, \partial x} - 4 \frac{\partial A}{\partial y} \frac{\partial^2 \phi}{\partial x \, \partial y} - 2 \frac{\partial^2 A}{\partial y^2} \frac{\partial \phi}{\partial x} + \frac{\partial^4 \phi}{\partial x^2 \, \partial y^2}. \quad (4.18)$$

In Eqs. (4.11) to (4.14), the mixed derivatives can be approximated by following difference formulae

$$(\partial^4\phi/\partial x^2 \, \partial y^2)_0 = [4\phi_0 - 2(\phi_1 + \phi_2 + \phi_3 + \phi_4) + (\phi_5 + \phi_6 + \phi_7 + \phi_8)]/h_1^2 h_2^2 + O(h^2) \quad (4.19)$$

$$(\partial^3\phi/\partial x^2 \, \partial y)_0 = [-2(\phi_2 - \phi_4) + \phi_5 + \phi_6 - \phi_7 - \phi_8]/2h_1^2 h_2 + O(h^2) \quad (4.20)$$

$$(\partial^3\phi/\partial x \, \partial y^2)_0 = [-2(\phi_1 - \phi_3) + \phi_5 - \phi_6 - \phi_7 + \phi_8]/2h_1 h_2^2 + O(h^2) \quad (4.21)$$

$$(\partial^2\phi/\partial x \, \partial y)_0 = (\phi_5 - \phi_6 + \phi_7 - \phi_8)/4h_1 h_2 + O(h^2) \quad (4.22)$$

and other derivatives by three-point central difference.

It is obvious that only nine points centered on a typical point (ih_1, jh_2) are involved in the perturbational value of the source term, which makes the overall scheme highly compact. As the perturbational value of the source term is of higher order, it is convenient to treat it as an element of the source term in iterative process. In Fig. 1, the points 5678 having to do with the perturbation of the source term are linked with dashed lines, so as to tell which are from the basic points 01234 of the perturbational h^4 accurate scheme.

5. PERTURBATIONAL h^4 ACCURATE EXPONENTIAL SCHEME IN THREE-DIMENSIONAL SPACE

Similar to the two-dimensional case, three-dimensional convective diffusion equation

$$2A \frac{\partial \phi}{\partial x} + 2B \frac{\partial \phi}{\partial y} + 2C \frac{\partial \phi}{\partial z} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} + S \quad (5.1)$$

has a perturbational h^4 accurate exponential scheme,

$$2\left[\frac{1}{h_1^2}\cosh(A_{pijk}h_1)+\frac{1}{h_2^2}\cosh(B_{pijk}h_2)+\frac{1}{h_3^2}\cosh(C_{pijk}h_3)\right]\phi_{ijk}$$
$$=\frac{1}{h_1^2}\left[\exp(A_{pijk}h_1)\,\phi_{i-1,j,k}+\exp(-A_{pijk}h_1)\,\phi_{i+1,j,k}\right]$$
$$+\frac{1}{h_2^2}\left[\exp(B_{pijk}h_2)\,\phi_{i,j-1,k}+\exp(-B_{pijk}h_2)\,\phi_{i,j+1,k}\right]$$
$$+\frac{1}{h_3^2}\left[\exp(-C_{pijk}h_3)\,\phi_{i,j,k-1}+\exp(-C_{pijk}h_3)\,\phi_{i,j,k+1}\right]+S_{pijk}, \tag{5.2}$$

where h_1, h_2, and h_3 denote grid sizes in the x, y, and z direction, and it is assumed that h_1, h_2, and h_3 are of the same order of magnitude, $O(h)$. In Eq. (5.2)

$$A_p=A+\Delta A_2\cdot h_1^2 \tag{5.3}$$
$$B_p=B+\Delta B_2\cdot h_2^2 \tag{5.4}$$
$$C_p=C+\Delta C_2\cdot h_3^2 \tag{5.5}$$
$$S_p=S+2(\Delta S_{2x}\cdot h_1^2+\Delta S_{2y}\cdot h_2^2+\Delta S_{2z}\cdot h_3^2) \tag{5.6}$$

and

$$\Delta A_2=\frac{1}{12}\left(2A\frac{\partial A}{\partial x}+\frac{\partial^2 A}{\partial x^2}\right)+O(h^2) \tag{5.7}$$
$$\Delta B_2=\frac{1}{12}\left(2B\frac{\partial B}{\partial y}+\frac{\partial^2 B}{\partial y^2}\right)+O(h^2) \tag{5.8}$$
$$\Delta C_2=\frac{1}{12}\left(2C\frac{\partial C}{\partial z}+\frac{\partial^2 C}{\partial z^2}\right)+O(h^2) \tag{5.9}$$
$$\Delta S_{2x}=\frac{1}{24}\left[2\left(A^2+2A\frac{\partial A}{\partial z}\right)S_x-2AP_x+Q_x\right]+O(h^2) \tag{5.10}$$
$$\Delta S_{2y}=\frac{1}{24}\left[2\left(B^2+2B\frac{\partial B}{\partial y}\right)S_y-2BP_y+Q_y\right]+O(h^2) \tag{5.11}$$
$$\Delta S_{2z}=\frac{1}{24}\left[2\left(C^2+2C\frac{\partial C}{\partial z}\right)S_z-2CP_z+Q_z\right]+O(h^2) \tag{5.12}$$

and

$$S_x=S+\left(-2B\frac{\partial\phi}{\partial y}+\frac{\partial^2\phi}{\partial y^2}\right)+\left(-2C\frac{\partial\phi}{\partial z}+\frac{\partial^2\phi}{\partial z^2}\right) \tag{5.13}$$
$$S_y=S+\left(-2C\frac{\partial\phi}{\partial z}+\frac{\partial^2\phi}{\partial z^2}\right)+\left(-2A\frac{\partial\phi}{\partial x}+\frac{\partial^2\phi}{\partial x^2}\right) \tag{5.14}$$
$$S_z=S+\left(-2A\frac{\partial\phi}{\partial x}+\frac{\partial^2\phi}{\partial x^2}\right)+\left(-2B\frac{\partial\phi}{\partial y}+\frac{\partial^2\phi}{\partial y^2}\right) \tag{5.15}$$
$$P_x=\frac{\partial S}{\partial x}-2B\frac{\partial^2\phi}{\partial x\,\partial y}-2\frac{\partial B}{\partial x}\frac{\partial\phi}{\partial y}+\frac{\partial^3\phi}{\partial x\,\partial y^2}-2C\frac{\partial^2\phi}{\partial x\,\partial z}-2\frac{\partial C}{\partial x}\frac{\partial\phi}{\partial z}+\frac{\partial^3\phi}{\partial x\,\partial z^2} \tag{5.16}$$
$$P_y=\frac{\partial S}{\partial y}-2C\frac{\partial^2\phi}{\partial y\,\partial z}-2\frac{\partial C}{\partial y}\frac{\partial\phi}{\partial z}+\frac{\partial^3\phi}{\partial y\,\partial z^2}-2A\frac{\partial^2\phi}{\partial y\,\partial x}-2\frac{\partial A}{\partial y}\frac{\partial\phi}{\partial x}+\frac{\partial^3\phi}{\partial y\,\partial x^2} \tag{5.17}$$
$$P_z=\frac{\partial S}{\partial z}-2A\frac{\partial^2\phi}{\partial z\,\partial x}-2\frac{\partial A}{\partial z}\frac{\partial\phi}{\partial x}+\frac{\partial^3\phi}{\partial z\,\partial x^2}-2B\frac{\partial^2\phi}{\partial z\,\partial y}-2\frac{\partial B}{\partial z}\frac{\partial\phi}{\partial y}+\frac{\partial^3\phi}{\partial z\,\partial y^2} \tag{5.18}$$
$$Q_x=\frac{\partial^2 S}{\partial x^2}-2B\frac{\partial^3\phi}{\partial x^2\,\partial y}-4\frac{\partial B}{\partial x}\frac{\partial^2\phi}{\partial x\,\partial y}-2\frac{\partial^2 B}{\partial x^2}\frac{\partial\phi}{\partial y}+\frac{\partial^4\phi}{\partial x^2\,\partial y^2}-2C\frac{\partial^3\phi}{\partial x^2\,\partial z}-4\frac{\partial C}{\partial x}\frac{\partial^2\phi}{\partial x\,\partial z}-2\frac{\partial^2 C}{\partial x^2}\frac{\partial\phi}{\partial z}+\frac{\partial^4\phi}{\partial x^2\,\partial z^2} \tag{5.19}$$
$$Q_y=\frac{\partial^2 S}{\partial y^2}-2A\frac{\partial^3\phi}{\partial y^2\,\partial x}-4\frac{\partial A}{\partial y}\frac{\partial^2\phi}{\partial x\,\partial y}-2\frac{\partial^2 A}{\partial y^2}\frac{\partial\phi}{\partial x}+\frac{\partial^4\phi}{\partial x^2\,\partial y^2}-2A\frac{\partial^3\phi}{\partial y^2\,\partial x}-4\frac{\partial A}{\partial y}\frac{\partial^2\phi}{\partial y\,\partial x}-2\frac{\partial^2 C}{\partial y^2}\frac{\partial\phi}{\partial x}+\frac{\partial^4\phi}{\partial y^2\,\partial x^2} \tag{5.20}$$
$$Q_z=\frac{\partial^2 S}{\partial z^2}-2A\frac{\partial^3\phi}{\partial z^2\,\partial x}-4\frac{\partial A}{\partial z}\frac{\partial^2\phi}{\partial z\,\partial x}-2\frac{\partial^2 A}{\partial z^2}\frac{\partial\phi}{\partial x}+\frac{\partial^4\phi}{\partial z^2\,\partial x^2}-2B\frac{\partial^3\phi}{\partial z^2\,\partial y}-4\frac{\partial B}{\partial z}\frac{\partial^2\phi}{\partial z\,\partial y}-2\frac{\partial^2 B}{\partial z^2}\frac{\partial\phi}{\partial y}+\frac{\partial^4\phi}{\partial z^2\,\partial y^2}. \tag{5.21}$$

Approximations of the derivatives in above equations are similar to those for the two-dimensional case.

6. NUMERICAL EXAMPLES

One- to three-dimensional model equations of fluid flow and a problem of natural convective heat transfer are solved with the present exponential schemes. The model equations are also solved with the compact h^4 scheme given by Dennis and Hudson [9] or the custamary h^2 central difference scheme, for comparison purposes. Results are compared with corresponding exact solutions or previous benchmark solutions. Simple line iterative successive over-relaxation procedures with suitable choice of relaxation parameter are used to solve the difference equations. The iterative process is repeated until

$$|\phi_{ij}^{(k+1)} - \phi_{ij}^{(k)}| \leqslant \varepsilon$$

for all grid points, where k is the iterative count, ε is determined as 10^{-10} for double precision arithmetic employed in solving two-dimensional model equation, or 10^{-5} for single precision arithmetic in other examples.

EXAMPLE 1. As a simple example of convective diffusion equation with solution involving abrupt change in function value, we consider the one-dimensional model equation of fluid flow, namely the well-known Burgers equation

$$u\frac{\partial u}{\partial x} = \frac{1}{\text{Re}}\frac{\partial^2 u}{\partial x^2} \tag{6.1}$$

with boundary conditions to give its solution as

$$u = \tanh[\text{Re}(1-2x)/4]. \tag{6.2}$$

In cases with large Reynolds number Re, this solution contains an abrupt change centred at the point $x = 0.5$, thus to model the nonlinear effects, such as the viscous boundary layer and the shock wave, of fluid flows.

The computational region $(0 < x < 1)$ is distributed uniformly with 20 grid points. Some typical numerical results are given in Fig. 2 and Table I, where U_a is the exact solution given by Eq. (6.2); U_b is the finite difference approximation given by the present h^2 accurate exponential scheme, U_c is the approximation by the present h^4 accurate exponential scheme, U_d is by Dennis and Hudson's h^4 scheme. It is shown that (i) for small Re (corresponding to small cell Reynolds-number Re_c) the numerical accuracy achieved by the present h^4 scheme is similar or identical with that by the Dennis and Hudson h^4 scheme, and the present h^4 scheme can give much more accurate results than the present h^2 scheme; (ii) for convection-dominated cases with large Re, both the present h^2 and h^4 schemes can resolve accurately the "viscous boundary layer"- or "shock wave"-like steep changes, while the Dennis and Hudson h^4 scheme gives only very poor or intolerable results, as is predicted in the introduction of this paper.

TABLE I

Solutions of the Burgers Equation (6.1)

$(x+h)/h$	U_a	U_b	U_c	U_d
		Re = 10		
7	0.7264	0.7289	0.7264	0.7264
8	0.5770	0.5798	0.5769	0.5769
9	0.3754	0.3779	0.3754	0.3754
10	0.1308	0.1318	0.1308	0.1308
11	−0.1308	−0.1318	−0.1308	−0.1318
12	−0.3754	−0.3779	−0.3754	−0.3754
13	−0.5770	−0.5798	−0.5769	−0.5769
14	−0.7264	−0.7289	−0.7264	−0.7264
		Re = 500		
7	1.0000	1.0000	1.0000	0.8128
8	1.0000	1.0000	1.0000	0.6972
9	1.0000	1.0000	1.0000	0.5148
10	1.0000	1.0000	1.0000	0.2270
11	−1.0000	−1.0000	−1.0000	−0.2270
12	−1.0000	−1.0000	−1.0000	−0.5148
13	−1.0000	−1.0000	−1.0000	−0.6972
14	−1.0000	−1.0000	−1.0000	−0.8128
		Re = 100,000		
7	1.0000	1.0000	1.0000	0.3710
8	1.0000	1.0000	1.0000	0.2653
9	1.0000	1.0000	1.0000	0.1594
10	1.0000	1.0000	1.0000	0.0532
11	−1.0000	−1.0000	−1.0000	−0.0532
12	−1.0000	−1.0000	−1.0000	−0.1594
13	−1.0000	−1.0000	−1.0000	−0.2653
14	−1.0000	−1.0000	−1.0000	−0.3710

EXAMPLE 2. The two-dimensional model equation for fluid flow, which is considered by Roscoe [10] and Dennis and Hudson [7–9], can be modified as

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - (2\sin y + \sin x)\cos x, \tag{6.3}$$

where

$$v = \sin x \cos y. \tag{6.4}$$

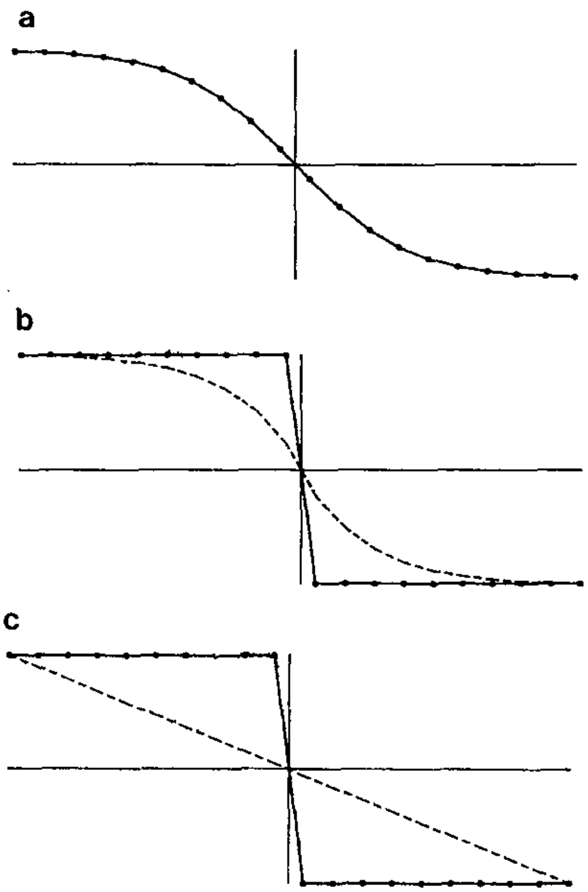

FIG. 2. Comparison of solutions of the Burgers equation (6.1). The real line stands for the exact solution, dashed line for the solution given by the compact h^4 scheme of Dennis and Hudson, dotted real line for the present exponential schemes: (a) Re = 10; (b) Re = 500; (c) = 100,000.

Exact solution of this equation is

$$u = -\cos x \sin y. \qquad (6.5)$$

The term on the right-hand side of Eq. (6.3) does not constitute all the possible pressure gradients for Eq. (6.3) to be truly representative of the momentum equation, but Eqs. (6.4) and (6.5) satisfy the equation of continuity. The solution is sought within the square region $0 < x < \pi$, $0 < y < \pi$, subject to boundary conditions for u and v calculated from Eqs. (6.5) and (6.4), respectively, on the square boundaries. To demonstrate numerically that our finite difference method of perturbation is h^4 accurate, two different square grid sizes (i) $h = \pi/10$ and (ii) $h = \pi/20$ are used in the calculation. In Table II we give details of the computed results for $u(0.7, y)$ for values of y/π from 0 to 0.5, where U_a is the exact solution given by Eq. (6.5), U_b is by the h^4 scheme with $h = \pi/20$; U_c is by the h^4 scheme, U_d is by the h^2 exponential scheme, U_e is by the h^2 central difference scheme, all with $h = \pi/10$; and E_r is the ratio of the error in

TABLE II

Solutions of the Two-Dimensional Model Equation (6.3)

x/π	y/π	U_a	U_b	U_c	U_d	U_e	E_r
0.7	0.1	0.1816356	0.1816368	0.1816541	0.1827	0.1830	15.4
0.7	0.2	0.3454915	0.3454934	0.3455220	0.3473	0.3478	16.1
0.7	0.3	0.4755283	0.4755308	0.4755699	0.4778	0.4786	16.6
0.7	0.4	0.5590170	0.5590201	0.5590679	0.5616	0.5625	16.4
0.7	0.5	0.5877852	0.5877886	0.5878399	0.5905	0.5914	16.1

the coarse mesh solution at a given point to the corresponding error in the fine mesh solution, which is approximately 16, as one would expect from an h^4 accurate scheme when the steplength is halved. A solution correct to four decimal accuracy is obtained using a grid size of $h = \pi/10$ by the h^4 accurate scheme.

EXAMPLE 3. Similar to two-dimensional case, we design a three-dimensional model equation of fluid flow

$$\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \\ -\cos x[2\sin y + 2\sin z \\ + \sin x(\sin y + \sin z)^2 \\ + \cos^2 y(\sin y + \sin z) \\ -\cos^2 z(\sin y - \sin x)], \end{aligned} \qquad (6.6)$$

where

$$v = \cos y(\sin x + \sin z) \qquad (6.7)$$

$$w = -\cos z(\sin y - \sin x). \qquad (6.8)$$

Exact solution of Eq. (6.6) is

$$u = -\cos x(\sin y + \sin z). \qquad (6.9)$$

The numerical solution is sought within the square region $0 < x < \pi$, $0 < y < \pi$, $0 < z < \pi$ with grid sizes $h_1 = h_2 = h_3 = \pi/10$. Typical results are given in Table III, where U_a is the exact solution, U_b is the numerical solution by the present h^4 accurate scheme, U_c is by the h^2 exponential scheme, U_d is by the h^2 reference scheme. Advantage of the present h^4 scheme over the present h^2 scheme and that of the present h^2 scheme over the h^2 reference scheme (the central difference scheme) can be appreciated.

EXAMPLE 4. As a final example we consider the problem of two-dimensional natural convective heat transfer in a square cavity defined by $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$, of which benchmark solutions have been given by de Vahl Davis [3]

TABLE III

Solutions of the Three-Dimensional Model Equation (6.6)

x/π	y/π	z/π	U_a	U_b	U_c	U_d
0.7	0.7	0.1	0.657163	0.657174	0.658629	0.660356
0.7	0.7	0.2	0.821018	0.821042	0.823179	0.826458
0.7	0.7	0.3	0.951055	0.951090	0.953437	0.958047
0.7	0.7	0.4	1.034543	1.034587	1.036927	1.042472
0.7	0.7	0.5	1.063312	1.063358	1.065671	1.071554

and a comparison solution has also been obtained by Dennis and Hudson [9], using their h^4 accurate approximations. In terms of dimensionless variables the governing equations can be written as

$$\nabla^2\psi = -\zeta \tag{6.10a}$$

$$\nabla^2\zeta = \mathrm{Pr}^{-1}\left(u\frac{\partial\zeta}{\partial x} + v\frac{\partial\zeta}{\partial y}\right) - \mathrm{Ra}\frac{\partial T}{\partial x} \tag{6.10b}$$

$$\nabla^2 T = u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}, \tag{6.10c}$$

where $u = \partial\psi/\partial y$, $v = -\partial\psi/\partial x$ are the two-dimensional velocity components in the x and y directions, Ra is the Rayleigh number, Pr is the Prandtl number, and operator $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$. If C denotes the unit square and n is the outward normal to it at any point, Eqs. (6.10) are to be solved within C, subject to the conditions

$$\psi = \partial\psi/\partial n = 0 \quad \text{on } C; \tag{6.11a}$$

$$T = 1 \quad \text{when} \quad x = 0, \qquad T = 0 \quad \text{when} \quad x = 1; \tag{6.11b}$$

$$\partial T/\partial y = 0 \quad \text{when} \quad y = 0, 1 \tag{6.11c}$$

The three equations (6.10) are each of the convective diffusion type and can thus be approximated by the perturbational h^4 accurate scheme discussed in Section 4. The term involving $\partial T/\partial x$ in Eq. (6.10b) is a forcing term coming from the solution of Eq. (6.10c) and must be approximated in Eq. (6.10b) correct to h^4 accuracy. According to Eqs. (4.13) and (4.14), the perturbational value of the source term has to do with $\partial T/\partial x$, $\partial^2 T/\partial x^2$, $\partial^3 T/\partial x^3$, $\partial^2 T/\partial x\,\partial y$, and $\partial^3 T/\partial x\,\partial y^2$, where $\partial T/\partial x$ and $\partial^2 T/\partial x^2$ can be approximated by normal three-point central difference, $\partial^3 T/\partial x\,\partial y$ and $\partial^3 T/\partial x\,\partial y^2$ by the difference formulae (4.22) and (4.21), leaving $\partial^3 T/\partial x^3$ approximated by

$$(\partial^3 T/\partial x^3)_{ij} = (T_{i+2,j} - 2T_{i+1,j} + 2T_{i-1,j} - T_{i-2,j})/2h^3 + O(h^2) \tag{6.12}$$

TABLE IV

Properties of Natural Convection in a Square Cavity ($\mathrm{Pr} = 0.71$, $\mathrm{Ra} = 10^3$)

h	$\lvert\psi_{\mathrm{mid}}\rvert$	$U_{\max}$ $y(x=0.5)$	$V_{\max}$ $x(y=0.5)$	Nu_0	$Nu_{\max}$ $y(x=0)$	$Nu_{\min}$ $y(x=0)$
1/10	1.1781	3.6695	3.7181	1.1168	1.4995	0.6930
		0.8125	0.1805		0.0693	1
1/20	1.1750	3.6510	3.6999	1.1170	1.5069	0.6919
		0.8135	0.1785		0.0825	1
1/30	1.1746	3.6501	3.6980	1.1172	1.5055	0.6915
		0.8131	0.1784		0.0860	1
BMS	1.174	3.649	3.697	1.117	1.505	0.692
		0.813	0.178		0.092	1

Note. BMS stands for the benchmark solution.

with respect to square grids. Formula (6.12) involves nodal values of five points along the x-line, in this way the nodal value of the first external point on the normal through the vertical boundaries $x = 0$ and $x = 1$ is required, which may be obtained using an h^5 accurate extrapolating formula. The boundary vorticity can be expressed by an h^4 accurate formula

$$\zeta_B = \frac{15}{23h^2}(8\psi_{I_1} - \psi_{I_2}) - \frac{1}{23}(16\zeta_{I_1} - 11\zeta_{I_2} + 2\zeta_{I_3}) + O(h^4) \tag{6.13}$$

given by Dennis and Hudson [9], where I_1, I_2, and I_3 denote the first three neighbouring internal points on the normal through the boundary B.

Solutions for $\mathrm{Pr} = 0.71$, $\mathrm{Ra} = 10^3$, 10^4, 10^5 with $h = 1/10$, 1/20, 1/30, using the perturbational h^4 accurate scheme are obtained. In each case the program performed one iteration of the vorticity field followed by n iterations of the stream-

TABLE V

Properties of Natural Convection in a Square Cavity ($\mathrm{Pr} = 0.71$, $\mathrm{Ra} = 10^4$)

h	$\lvert\psi_{\mathrm{mid}}\rvert$	$U_{\max}$ $y(x=0.5)$	$V_{\max}$ $x(y=0.5)$	Nu_0	$Nu_{\max}$ $y(x=0)$	$Nu_{\min}$ $y(x=0)$
1/10	5.0705	16.2835	19.3567	2.3333	3.6927	0.5915
		0.8215	0.1280		0.1697	1
1/20	5.0733	16.1850	19.6555	2.2337	3.5075	0.5897
		0.8233	0.1195		0.1401	1
1/30	5.0735	16.1809	19.6297	2.2361	3.5180	0.5855
		0.8233	0.1191		0.1435	1
BMS	5.071	16.178	19.617	2.238	3.528	0.586
		0.823	0.119		0.143	1

TABLE VI

Properties of Natural Convection in a Square Cavity (Pr = 0.71, Ra = 10^5)

h	$\|\psi_{mid}\|$	U_{max} $y(x=0.5)$	V_{max} $x(y=0.5)$	Nu_0	Nu_{max} $y(x=0)$	Nu_{min} $y(x=0)$
1/10	9.4156	35.2363	50.2577	4.8730	8.7502	0.7409
		0.8560	0.0763		0.2005	1
1/20	9.2056	34.9781	69.0465	4.7053	8.3506	0.7289
		0.8530	0.0672		0.1023	1
1/30	9.1105	34.6977	68.7055	4.5220	7.8126	0.7278
		0.8550	0.0669		0.0801	1
BMS	9.111	34.730	68.590	4.509	7.717	0.729
		0.855	0.066		0.081	1

function and temperature fields. In general we set $n = 1$; however, for the cases with Ra = 10^5 we increased n to around 15 and introduced a relaxation factor of 0.45 to ensure convergence of the iterative procedure. Typical results are listed in Table IV–VI along with the benchmark solution given by de Vahl Davis [3] for comparison purposes, where we give (i) the magnitude of the streamfunction at the mid-point of the cavity; (ii) the maximum value of u on the vertical mid-plane, together with its location; (iii) the maximum value of v on the horizontal mid-plane, together with its location; (iv) the average Nusselt number Nu_0, on the vertical boundary at $x = 0$; (v) the maximum and minimum values of the local Nusselt number on $x = 0$, together with their locations. The maximum values (and their locations) referred to above are evaluated with an h^4 accurate interpolating polynomial. The local Nusselt numbers on the boundary at $x = 0$ are estimated by a numerical differentiation formula of fourth order. The benchmark solution of de Vahl Davis [3] were obtained by using mesh refinement and extrapolation in conjunction with a second-order method and are claimed to be very accurate.

To obtain a clearer assessment of our $O(h^4)$ results we give, in Table VII, the percentage differences between various properties of our solutions and those of the benchmark solutions. De Vahl Davis [3] has estimated that the percentage errors of his benchmark solution for Ra = 10^3, 10^4, and 10^5 are no more than 0.1, 0.2, and 0.3, respectively. From Table VII we see that our results based on $h = \frac{1}{30}$ are well within these tolerances for all Rayleigh numbers considered. It is relevant to note that the extrapolated bench-mark solutions were based on two $O(h^2)$ solutions obtained with (i) $h = \frac{1}{20}, \frac{1}{40}$ when Ra = 10^3, 10^4 and (ii) $h = \frac{1}{40}, \frac{1}{80}$ when Ra = 10^5. Thus the perturbational h^4 accurate scheme gives results of comparable accuracy on a single comparatively coarse mesh.

TABLE VII

Percentage Differences between Various Properties of the Benchmark Solutions and Those of the Present Perturbational Fourth Accurate Solutions

Ra	h	$\|\psi_{mid}\|$	U_{max}	V_{max}	Nu_0
10^3	1/10	0.35	0.56	0.57	0.02
	1/20	0.09	0.05	0.08	0.00
	1/30	0.05	0.03	0.03	0.02
10^4	1/10	0.01	0.65	0.33	4.26
	1/20	0.05	0.04	0.20	0.19
	1/30	0.05	0.02	0.06	0.08
10^5	1/10	0.34	1.46	26.73	8.07
	1/20	1.04	0.71	0.66	4.35
	1/30	1.01	0.10	0.17	0.29

ACKNOWLEDGMENT

The authors are in debt to Professor Chen Yao-Song of the Department of Mechanics, Peking University, for his precious advice on some parts of this paper.

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